Supplementary Topic: Poisson races

Race problem. We repeat an experiment with probability $p \in (0,1)$ of success (and probability $q = 1-p$ of failure). For some positive integers m and n, what is the probability that we get m successes before a total of n failures?

Solution. We need to realize that the event in problem is equivanlent to having at least m successes in the first $m+n-1$ experiments. If it happens, there are at most $n-1$ failures before the mth success. On the other hand, if we just have $m-1$ or fewer successes in the first $m+n-1$ experiments, then there are at least n failures happen.

Hence the probability of that event is

$$\sum_{k=m}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}. \tag{1}$$

In the last tutorial, we discuss about the independent sum and decomposition of Poisson processes. As an application, we take the kind of 'races' above into consideration.

Theorem 3. Given a Poisson process $X_1(s)$ of type-1 arrivals with rate $\lambda > 0$, and an independent Poisson process $X_2(s)$ of type-2 arrivals with rate $\mu > 0$. Then the probability that one get m type-1 arrivals before a total of n type-2 arrivals is

$$\sum_{k=m}^{m+n-1} \binom{m+n-1}{k} \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{m+n-1-k}. \tag{2}$$

Proof. Let $X(s) = X_1(s) + X_2(s)$. Then $X(s)$ is a Poisson process with rate $\lambda + \mu$ (see the "independent sum" part). We can realize $X_1(s)$ and $X_2(s)$ as being constructed by starting with $X(s)$ and repeating an experiment with probability $p = \lambda/(\lambda+\mu)$ of success to decide the type of arrivals (see the "decomposition" part): be type-1 if the experiment succeeds and be type-2 if it fails. Hence we see from (5) that the probability of interest is given by (6).

Example 1. Ellen catches fish at times of a Poisson process with rate 2 per hour, 40% of the fish are salmon, while 60% of the fish are trout.

(a) What is the probability that she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours?

(b) What is the probability that she will catch 4 trout before she catches 3 salmon?

Solution. Let $X_1(s)$ and $X_2(s)$ be the numbers of salmon and trout caught by Ellen in s hours. Then $X_1(s)$ and $X_2(s)$ are two independent Poisson processes with rate $2 \times 40\% = 0.8$ and $2 \times 60\% = 1.2$ respectively (see the "decomposition" part).

(a) Note that $X_1(2.5) \sim \text{Poi}(2)$, $X_2(2.5) \sim \text{Poi}(3)$. Hence the probability in the question is

$$P(X_1(2.5) = 1, X_2(2.5) = 2) = e^{-2}\frac{2^1}{1!} \cdot e^{-3}\frac{3^2}{2!} = 9e^{-5} \doteq 0.0606.$$

(b) This is a Poisson race problem. The probability in the question is

$$\begin{aligned}
&\sum_{k=4}^{4+3-1} \binom{4+3-1}{k} \left(\frac{1.2}{1.2+0.8}\right)^k \left(\frac{0.8}{1.2+0.8}\right)^{4+3-1-k} \\
=\ & \binom{6}{4}(0.6)^4(0.4)^2 + \binom{6}{5}(0.6)^5(0.4)^1 + \binom{6}{6}(0.6)^6 \\
=\ & 0.54432.
\end{aligned}$$

Example 2. Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that the two tellers have exponential service times with means 3 and 6 minutes.

(a) What is the expected total amount of time for Carol to complete her business?

(b) What is the expected total time until the last of the three customers leaves?

(c) What is the probability Carol is the last one to leave?

We have two ways to solve these problems. The first solution is to solve it directly while the second solution is an application of Theorems 1 & 2 in tutorial 10 (It is more interesting!).

Solution 1. Let A, B, C be the serving time of Alice, Betty, and Carol respectively. Then $A \sim \text{Exp}(1/3)$ and $B \sim \text{Exp}(1/6)$. Moreover, we know

$$C \sim \begin{cases} A \sim \text{Exp}(1/3), & \text{if } A \le B, \\ B \sim \text{Exp}(1/6), & \text{if } A > B. \end{cases}$$

(a) The waiting time for Carol is $\min\{A, B\}$. Hence the expected total amount of time for Carol to complete is

$$\begin{aligned} \mathbb{E}(\min\{A,B\} + C) &= \mathbb{E}(\min\{A,B\}) + \mathbb{E}[C] \\ &= \mathbb{E}(\min\{A,B\}) + \mathbb{E}(A)\mathbb{P}(A \le B) + \mathbb{E}(B)\mathbb{P}(A > B) \\ &= \frac{1}{1/3+1/6} + 3 \cdot \frac{1/3}{1/3+1/6} + 6 \cdot \frac{1/6}{1/3+1/6} \\ &= 2+2+2 = 6 \text{ minutes.} \end{aligned}$$

(b) The expected total time until the last of the three customers leaves is

$$\begin{aligned} & E(\max\{A, B, \min\{A,B\} + C\}) \\ = \;& E(\min\{A,B\}) + E(\max\{A - \min\{A,B\}, B - \min\{A,B\}, C\}) \\ = \;& E(\min\{A,B\}) + E(\max\{B-A, C\} \mid B > A)P(B > A) + \\ & E(\max\{A-B, C\} \mid A > B)P(A > B) \\ & \text{(by the lack of memory property)} \\ = \;& E(\min\{A,B\}) + E(\max\{B,A\})P(B > A) + E(\max\{A,B\})P(A > B) \\ = \;& E(\min\{A,B\}) + E(\max\{B,A\}) \\ = \;& EA + EB \\ = \;& 9 \text{ minutes.} \end{aligned}$$

(c) The probability that Carol is the last one to leave is

$$\begin{aligned} & P(\max\{A, B, \min\{A,B\} + C\} = \min\{A,B\} + C) \\ = \;& P(\min\{A,B\} + C > \max\{A,B\}) \\ = \;& P(\min\{A,B\} + C > \max\{A,B\} \mid A < B)P(A < B) + \\ & P(\min\{A,B\} + C > \max\{A,B\} \mid A > B)P(A > B) \\ = \;& P(C > B - A \mid B > A)P(B > A) + P(C > A - B \mid A > B)P(A > B) \\ = \;& P(A > B)P(B > A) + P(B > A)P(A > B) \\ = \;& 2P(B > A)P(A > B) \\ = \;& 2 \cdot \frac{1/3}{1/3+1/6} \cdot \frac{1/6}{1/3+1/6} \\ = \;& \frac{4}{9}. \end{aligned}$$

Solution 2. Suppose that there are infinitely many customers in the bank waiting for service. Consider this infinite service sequence, then the numbers of customers served by each teller in t minutes is a Possion process $X_i(t)$, $i = 1, 2$, with rate $1/3$ and $1/6$ respectively. By Theorem 1, the sum $X(t) = X_1(t) + X_2(t)$ is a Poisson process with rate $1/3 + 1/6 = 1/2$. By Theorem 2 (conversed), each arrival in $X(t)$ has probability $(1/3)/(1/2) = 2/3$ to be in $X_1(t)$ (type-1), and probability $1 - 2/3 = 1/3$ to be in $X_2(t)$ (type-2). Let τ_n be the waiting time for the n-th arrival in $X(t)$, and let A_n be the event that the n-th arrival is of type-1. So $E(\tau_n) = 2n$ and $P(A_n) = 2/3$ for each $n \geq 1$. The first arrival in type-1 (or type-2) means that Alice (or Betty) completes her business. If A_1 (or A_1^c) happens, then the second arrival in type-1 (or type-2) means that Carol completes her business. Let C be the serving time of Carol.

(a) Using the same argument as in Solution 1, the expected total amount of time for Carol to complete is

$$E(\tau_1 + C) = E(\tau_1) + E(C \mid A_1)P(A_1) + E(C \mid A_1^c)P(A_1^c) = 2 + 3(2/3) + 6(1/3) = 6.$$

(b) Let Y be the extra serving time for the last person after time τ_2. Note that no matter who is the first person to leave, the remaining two must be served by different tellers (*why?*). As the exponentially distributed random variable is memoryless, we have $E(Y \mid A_2) = 6$ (the mean serving time of the second teller) and $E(Y \mid A_2^c) = 3$ (the mean serving time of the first teller). Then the expected total time until the last of the three customers leaves is

$$E(\tau_2 + Y) = E(\tau_2) + E(Y \mid A_2)P(A_2) + E(Y \mid A_2^c)P(A_2^c) = 4 + 6(2/3) + 3(1/3) = 9.$$

(c) The probability that Carol is the last one to leave is

$$P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (2/3)(1/3) + (1/3)(2/3) = 4/9.$$